

Some Algorithmic Results on Restrained Domination in Graphs

Arti Pandey^{*1} and B. S. Panda^{†2}

¹Department of Computer Science and Engineering, IIIT Guwahati
Ambari, G. N. B. Road, Guwahati 781001, INDIA

²Department of Mathematics, Indian Institute of Technology Delhi
Hauz Khas, New Delhi 110016, INDIA

June 9, 2016

Abstract

A set $D \subseteq V$ of a graph $G = (V, E)$ is called a restrained dominating set of G if every vertex not in D is adjacent to a vertex in D and to a vertex in $V \setminus D$. The MINIMUM RESTRAINED DOMINATION problem is to find a restrained dominating set of minimum cardinality. Given a graph G , and a positive integer k , the RESTRAINED DOMINATION DECISION problem is to decide whether G has a restrained dominating set of cardinality at most k . The RESTRAINED DOMINATION DECISION problem is known to be NP-complete for chordal graphs. In this paper, we strengthen this NP-completeness result by showing that the RESTRAINED DOMINATION DECISION problem remains NP-complete for doubly chordal graphs, a subclass of chordal graphs. We also propose a polynomial time algorithm to solve the MINIMUM RESTRAINED DOMINATION problem in block graphs, a subclass of doubly chordal graphs. The RESTRAINED DOMINATION DECISION problem is also known to be NP-complete for split graphs. We propose a polynomial time algorithm to compute a minimum restrained dominating set of threshold graphs, a subclass of split graphs. In addition, we also propose polynomial time algorithms to solve the MINIMUM RESTRAINED DOMINATION problem in cographs and chain graphs. Finally, we give a new improved upper bound on the restrained domination number, cardinality of a minimum restrained dominating set in terms of number of vertices and minimum degree of graph. We also give a randomized algorithm to find a restrained dominating set whose cardinality satisfy our upper bound with a positive probability.

Keywords: Domination, Restrained domination, NP-completeness, Chordal graphs, Doubly chordal graphs, Threshold graphs, Cographs, Chain graphs.

1 Introduction

For a graph $G = (V, E)$, the sets $N_G(v) = \{u \in V(G) \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and *closed neighborhood* of a vertex v , respectively. A vertex v of a graph G is said to *dominate* a vertex w if $w \in N_G[v]$. A set $D \subseteq V$ is a *dominating set* of G if every vertex of G is dominated by at least one vertex in D . The MINIMUM DOMINATION problem is to find a dominating

^{*}artipandey2305@gmail.com

[†]bspanda@maths.iitd.ac.in

set of minimum cardinality. Given a graph G , and a positive integer k , the DOMINATION DECISION problem is to decide whether G has a dominating set of cardinality at most k . The *domination number* of a graph G , denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G . The concept of domination and its variations are widely studied as can be seen in [17, 18].

A dominating set D is called a *restrained dominating set* if every vertex not in D is adjacent to some other vertex in $V \setminus D$. The *restrained domination number* of a graph G , denoted by $\gamma_r(G)$, is the cardinality of a minimum restrained dominating set of G . The concept of restrained domination was introduced by Telle and Proskurowski [24] in 1997, albeit indirectly, as a vertex partitioning problem. The restrained domination has been widely studied, see [3, 8, 9, 12, 13, 14, 15, 16, 19, 23, 26]. An application of the concept of restrained domination is that of prisoners and guards. Each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. Note that position of each prisoner is observed by a guard (to effect security) while position of each prisoner is also seen by at least one other prisoner (to protect the rights of prisoners). To minimize the cost, we want to place as few guards as possible. The restrained domination problem and its decision version are as follows:

MINIMUM RESTRAINED DOMINATION (MRD) problem

Instance: A graph $G = (V, E)$.

Solution: A restrained dominating set D_r of G .

Measure: Cardinality of D_r .

RESTRAINED DOMINATION DECISION (RDD) problem

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Does there exist a restrained dominating set D_r of G such that $|D_r| \leq k$?

In the algorithmic graph theory, we are mainly interested in the borderline between polynomial time and NP-completeness for a given graph problem. One hierarchy of graph classes is: trees \subset block graphs \subset doubly chordal graphs \subset chordal graphs. In this hierarchy polynomial-time algorithm for restrained domination problem is known only for trees, while it is known to be NP-complete for chordal graphs. Here we emphasize on the gap of complexity between block graphs and doubly chordal graphs. We prove that the RESTRAINED DOMINATION DECISION problem is NP-complete for doubly chordal graphs and present a dynamic programming based polynomial time algorithm to compute the cardinality of a minimum restrained dominating set for block graphs. We also study the MINIMUM RESTRAINED DOMINATION problem on threshold graphs, cographs, and chain graphs.

It is also interesting to see whether there exists graph classes where domination and restrained domination problems differ in complexity. The MINIMUM DOMINATION problem is polynomial time solvable for doubly chordal graphs [2], but here we prove that the RESTRAINED DOMINATION DECISION problem is NP-complete for this graph class. On the other hand, we propose a graph class, where the MINIMUM RESTRAINED DOMINATION problem is easily solvable, but the DOMINATION DECISION problem is NP-complete. Next, we give a new upper bound on the restrained domination number using probabilistic approach. We also give a randomized algorithm to find a restrained dominating set of a graph whose expected cardinality satisfy the new upper bound.

The paper is organized as follows. In Section 2, some pertinent definitions and some preliminary results are discussed. In Section 3, we have shown that the RESTRAINED DOMINATION DECISION

problem is NP-complete for doubly chordal graphs. In Section 4, we have shown the graph classes where the MINIMUM DOMINATION problem and the MINIMUM RESTRAINED DOMINATION problem differ in complexity. In Section 5, we proposed a dynamic programming based algorithm to find a minimum restrained dominating set of block graphs. In Section 6, we studied the MINIMUM RESTRAINED DOMINATION problem in threshold graphs. In Section 7, we studied the MINIMUM RESTRAINED DOMINATION problem in cographs. In Section 8, we studied the MINIMUM RESTRAINED DOMINATION problem in chain graphs. In Section 9, we studied a new upper bound on the restrained domination number of a graph, and we also proposed a randomized algorithm to find a restrained dominating set, whose expected cardinality satisfy the new upper bound. In Section 10, we conclude the paper.

2 Preliminaries

For a graph $G = (V, E)$, the *degree* of a vertex v is $|N_G(v)|$ and is denoted by $d_G(v)$. If $d_G(v) = 1$, then v is called a *pendant vertex*. For a set $S \subseteq V$ of the graph $G = (V, E)$, the subgraph of G induced by S is defined as $G[S] = (S, E_S)$, where $E_S = \{xy \in E | x, y \in S\}$. If $G[C]$, $C \subseteq V$, is a complete subgraph of G , then C is called a *clique* of G . A graph $G = (V, E)$ is said to be *bipartite* if $V(G)$ can be partitioned into two disjoint sets X and Y such that every edge of G joins a vertex in X to a vertex in Y , and such a partition (X, Y) of V is called a *bipartition*. A bipartite graph with bipartition (X, Y) of V is denoted by $G = (X, Y, E)$. A graph G is said to be a *chordal graph* if every cycle in G of length at least four has a chord, i.e., an edge joining two non-consecutive vertices of the cycle. A chordal graph $G = (V, E)$ is a *split graph* if V can be partitioned into two sets I and C such that C is a clique and I is an independent set. A vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G[v]$ is a clique of G . An ordering $\alpha = (v_1, v_2, \dots, v_n)$ is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \dots, v_n\}]$ for all i , $1 \leq i \leq n$. We have the following characterization for chordal graphs.

Theorem 2.1 ([10]). *A graph G has a PEO if and only if G is chordal.*

A vertex $u \in N_G[v]$ is a *maximum neighbor* of v in G if $N_G[w] \subseteq N_G[u]$ for all $w \in N_G[v]$. A vertex v in G is called *doubly simplicial* if it is a simplicial vertex and it has a maximum neighbor in G . An ordering $\sigma = (v_1, v_2, \dots, v_n)$ of V is a *doubly perfect elimination ordering* (DPEO) if v_i is a doubly simplicial vertex in the induced subgraph $G[\{v_i, v_{i+1}, \dots, v_n\}]$ for each i , $1 \leq i \leq n$. A graph is *doubly chordal* if it admits a doubly perfect elimination ordering (DPEO) [1].

In this paper, we only consider simple connected graphs with at least two vertices unless otherwise mentioned specifically.

We have the following straightforward observation for any restrained dominating set of a graph.

Observation 2.1. *Let G be a graph and D be any restrained dominating set of G . If P denotes the set of all pendant vertices in G , then $P \subseteq D$.*

3 Restrained domination in doubly chordal graphs

To show that the RESTRAINED DOMINATION DECISION problem is NP-complete, we need to use a well known NP-complete problem, called Exact Cover by 3-Sets (X3C) [11], which is defined as follows:

Exact Cover By 3-Sets (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection \mathcal{C} of 3-element subsets of X .

QUESTION: Does \mathcal{C} contain an exact cover for X , that is, a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element in X occurs in exactly one member of \mathcal{C}' ?

Theorem 3.1. *The RDD problem is NP-Complete for doubly chordal graphs.*

Proof. Clearly, the RDD problem is in NP. To show that it is NP-complete, we establish a polynomial time reduction from Exact Cover by 3-Sets (X3C). Let $X = \{x_1, x_2, \dots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of X3C.

We construct the graph $G = (V, E)$ and a positive integer k , as in instance of the RDD problem in the following way:

$$V = \{x_1, x_2, \dots, x_{3q}\} \cup \{c_1, c_2, \dots, c_m\} \cup \{w_1, w_2, \dots, w_q\} \cup \{z_1, z_2, \dots, z_q\} \cup \{r\},$$

$$E = \{x_i c_j | x_i \in C_j, 1 \leq i \leq 3q, 1 \leq j \leq m\} \cup \{c_i c_j | 1 \leq i < j \leq m\} \cup \{r x_i | 1 \leq i \leq 3q\} \cup \{r c_i | 1 \leq i \leq m\} \cup \{r w_i | 1 \leq i \leq q\} \cup \{w_i z_i | 1 \leq i \leq q\}, \text{ and } k = 2q.$$

The graph G is a doubly chordal graph as $(x_1, \dots, x_{3q}, c_1, \dots, c_m, z_1, z_2, \dots, z_q, w_1, w_2, \dots, w_q, r)$ is a DPEO of G . The construction of the graph $G = (V, E)$ associated with an instance of X3C, where $X = \{x_1, x_2, \dots, x_6\}$ and $\mathcal{C} = \{C_1 = (x_1, x_4, x_6), C_2 = (x_1, x_2, x_5), C_3 = (x_2, x_3, x_5), C_4 = (x_2, x_4, x_6), C_5 = (x_3, x_5, x_6)\}$ is shown in figure 1.

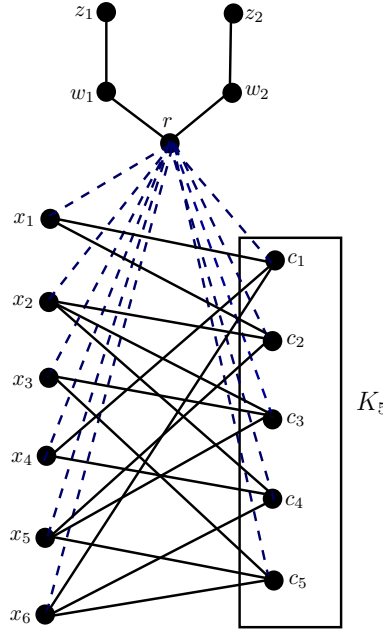


Figure 1: An Illustration to the construction of doubly chordal graph

Now we show that X has an exact cover $\mathcal{C}' \subseteq \mathcal{C}$ if and only if G has a restrained dominating set of cardinality at most k .

Suppose that X has an exact cover \mathcal{C}' . Then $\{c_j | C_j \in \mathcal{C}'\} \cup \{z_1, z_2, \dots, z_q\}$ is a restrained dominating set of cardinality $2q$.

Conversely, suppose that D is a restrained dominating set of G of cardinality at most $2q$. Then by Observation 2.1, all the pendant vertices of G must belong to D . Hence $\{z_1, z_2, \dots, z_q\} \subseteq D$. Next, we show that $r \notin D$. Since if $r \in D$, then the set $\{w_1, w_2, \dots, w_q\} \subseteq D$, and the cardinality of the set D must be at least $2q + 1$, which is not true. Therefore $r \notin D$.

Now define $D' = D \setminus (\{z_1, z_2, \dots, z_q\})$, and $X = \{x_1, x_2, \dots, x_{3q}\}$. Then $|D'| \leq q$, and all the vertices of X are dominated by D' . Since $N_G[X] = X \cup \{c_1, c_2, \dots, c_m\}$, all the $3q$ vertices of X are dominated by at most q vertices of $N_G[X]$. If for some i , $1 \leq i \leq n$, $x_i \in D$, then it dominates only a single vertex of X (x_i itself), and if for some j , $1 \leq j \leq m$, $c_j \in D$, then it dominates 3 vertices of X . But to dominate all the $3q$ vertices of X by at most q vertices of D' , each vertex in D' must dominate at least 3 vertices of X . Hence $X \cap D' = \emptyset$ and $|D' \cap \{c_1, c_2, \dots, c_m\}| = q$. This implies that $C' = \{C_j \mid c_j \in D'\}$ is an exact cover of \mathcal{C} .

Hence, the RDD problem is NP-complete for doubly chordal graphs. \square

4 Complexity difference in domination and restrained domination

In this section, we construct a class of graphs, for which the MINIMUM RESTRAINED DOMINATION problem is easily solvable, but the decision version of the domination problem is NP-complete.

Definition 4.1 (GP graph). A graph $G = (V_G, E_G)$ is said to be GP graph if it can be constructed from a general graph $H = (V_H, E_H)$, where $V_H = \{v_1, v_2, \dots, v_n\}$ in the following way: for each vertex v_i of H , add a path v_i, x_i, y_i, z_i of length 3.

Formally, $V_G = V_H \cup \{x_i, y_i, z_i \mid 1 \leq i \leq n\}$ and $E_G = E_H \cup \{v_i x_i, x_i y_i, y_i z_i \mid 1 \leq i \leq n\}$.

Theorem 4.1. Let G be a GP graph constructed from a general graph $H = (V_H, E_H)$, where $V_H = \{v_1, v_2, \dots, v_n\}$, by taking a path v_i, x_i, y_i, z_i of length 3, corresponding to each vertex $v_i \in V_H$. Then $\gamma_r(G) = 2n$ and $V_H \cup \{z_i \mid 1 \leq i \leq n\}$ is a restrained dominating set of G .

Proof. It is easy to observe that $V_H \cup \{z_i \mid 1 \leq i \leq n\}$ is a restrained dominating set of G . Hence $\gamma_r(G) \leq 2n$.

Now consider a restrained dominating set, say D_r of G . Then D_r must contain all the pendant vertices of G . Hence $\{z_i \mid 1 \leq i \leq n\} \subseteq D_r$. Now, to dominate x_i , at least one vertex from the set $\{v_i, x_i, y_i\}$ must belong to D_r , for each i , $1 \leq i \leq n$. This implies that $|D_r| \geq 2n$ and this completes the proof of the theorem. \square

The following theorem directly follows from the Theorem 4.1.

Theorem 4.2. A minimum dominating set of a GP graph can be computed in linear time.

Lemma 4.1. Let G be a GP graph constructed from a general graph $H = (V_H, E_H)$, where $V_H = \{v_1, v_2, \dots, v_n\}$, by taking a path v_i, x_i, y_i, z_i of length 3, corresponding to each vertex $v_i \in V_H$. Then H has a dominating set of cardinality at most k if and only if G has a dominating set of cardinality at most $n + k$.

Proof. Let D' be a dominating set of H of cardinality k . Then $D' \cup \{y_i \mid 1 \leq i \leq n\}$ is a dominating set of G of cardinality $n + k$.

Conversely, suppose that D is a dominating set of G of cardinality $n + k$. Then, either the pendant vertex z_i or the vertex adjacent to pendant vertex, that is, y_i must belong to D . Define $D' = D \setminus (\{y_1, y_2, \dots, y_n\} \cup \{z_1, z_2, \dots, z_n\})$. Then $|D'| \leq |D| - n$. Now, for each i , $1 \leq i \leq n$, if $x_i \in D'$, we update D' as $D' = (D' \setminus \{x_i\}) \cup \{v_i\}$. Clearly D' is a dominating set of H and $|D'| \leq |D| - n$. Hence D' is a dominating set of H of cardinality at most k . \square

We already have the following result for the decision version of the domination problem.

Theorem 4.3. [11] *The DOMINATION DECISION problem is NP-complete for general graphs.*

The following theorem directly follows from the Lemma 4.1 and Theorem 4.3.

Theorem 4.4. *The DOMINATION DECISION problem is NP-complete for GP graphs.*

5 Restrained domination in block graphs

A vertex $v \in V$ of a graph $G = (V, E)$ is called a *cut vertex* of G if $G \setminus \{v\}$, the subgraph of G obtained after removing the vertex v and all the edges incident on v , becomes disconnected. A maximal connected induced subgraph with no cut vertex is called a *block* of G . The intersection of two blocks contains at most one vertex. A vertex belongs to the intersection of two or more blocks if and only if it is a cut-vertex of the graph. A graph G is called a *block graph* if all the blocks of G are complete graphs. A block graph can be represented by a tree like decomposition structure, called *cut-tree*. The cut-tree, denoted by $T_{CG}(V^{CG}, E^{CG})$, of a block graph $G(V, E)$ with k blocks BC_1, BC_2, \dots, BC_k and l cut vertices v_1, v_2, \dots, v_l is defined in the following way:

$V^{CG} = \{BC_1, BC_2, \dots, BC_k, v_1, v_2, \dots, v_l\}$,
and $E^{CG} = \{(BC_i, v_j) | v_j \in V(BC_i), 1 \leq i \leq k, 1 \leq j \leq l\}$.

The cut-tree of a block graph G can be constructed in linear time using depth-first search method. For any block BC_i , define $B_i = \{v \in V(BC_i) | v \text{ is not a cut vertex}\}$. Now we can refine the cut-tree $T_{CG} = (V^{CG}, E^{CG})$ as $T_G = (V^G, E^G)$, where $V^G = \{(1, B_1), (2, B_2), \dots, (k, B_k), v_1, v_2, \dots, v_l\}$ and $E^G = \{((i, B_i), v_j) | v_j \in V(BC_i), 1 \leq i \leq k, 1 \leq j \leq l\}$. Each (i, B_i) is called a *block-vertex*. Note that one or more B_i may be empty. So we have used (i, B_i) instead of B_i . However, in the rest of the paper we will use B_i for (i, B_i) unless otherwise mentioned explicitly. Note that every vertex in the refined cut-tree is either a cut vertex or block vertex. A block graph G and the corresponding refined cut-tree are shown in Fig. 2. We consider the refined cut-tree $T_G = (V^G, E^G)$ of the block graph G , as input of our problem. Now, first we prove the following lemma.

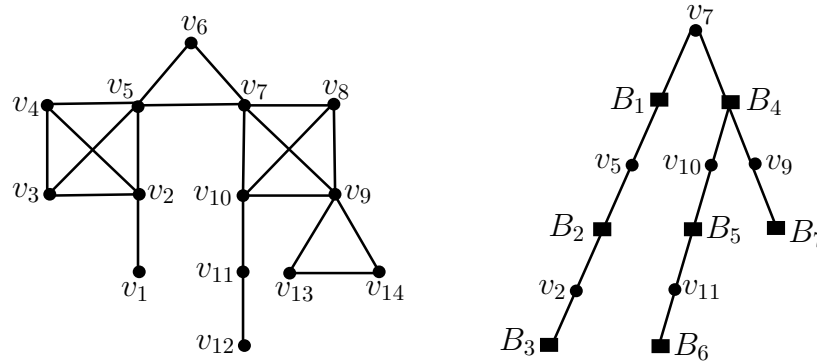


Figure 2: A block graph G and the corresponding refined cut-tree

Lemma 5.1. *Let $G = (V, E)$ be a block graph with at least three vertices. Then, every minimum restrained dominating set of G contains at most one vertex from each B_i .*

Proof. On the contrary, let D be a minimum cardinality restrained dominating set containing two vertices, say u and v , from some B_j (Note that D will not contain more than two vertices from B_j ; otherwise

$(D \setminus B_j) \cup \{u\}$ would have been a restrained dominating set of G with smaller cardinality than that of D contradicting the minimality of D).

Note that $N_G[u] = N_G[v]$. If $|B_j| \geq 3$, then let $x \in B_j \setminus \{u, v\}$. Now $D \setminus \{u\}$ is a restrained dominating set of G contradicting the minimality of D . Hence $|B_j| = 2$. Since G has at least three vertices, block BC_j contains one or more cut vertices. If D contains a cut vertex of BC_j , then define $D' = D \setminus \{u, v\}$ else define $D' = D \setminus \{u\}$. Then D' is also a restrained dominating set of G , and $|D'| < |D|$, which is a contradiction to the minimality of D . Hence D contains at most one vertex of B_j .

This completes the proof of our lemma. \square

5.1 Dynamic programming approach

Let $G = (V, E)$ be a block graph with at least three vertices. If $G = (V, E)$ has exactly one block, then G is a complete graph and $\{v\}, v \in V(G)$ is a minimum cardinality restrained dominating set of G . So assume that G be a connected block graph with at least two blocks, and $T^G = (V^G, E^G)$ be the refined cut-tree of G . We make T^G a rooted tree rooted at a cut vertex c of G . Consider the refined rooted cut-tree T_c^G rooted at c , corresponding to block graph G . We define the following parameters:

$A_c(G) = \text{Min} \{|D| : c \in D \text{ and } D \text{ is a restrained dominating set of } G\}$.

$B_c(G) = \text{Min} \{|D| : c \notin D \text{ and } D \text{ is a restrained dominating set of } G\}$.

Since a minimum restrained dominating set of G either contains the cut vertex c or does not contain the cut vertex c . Hence we have the following straightforward result.

Observation 5.1. $\gamma_r(G) = \text{Min}(A_c(G), B_c(G))$.

These parameters can be computed in a bottom up approach using the refined cut-tree rooted at the cut vertex c .

5.2 Parameters to be computed at a cut vertex in the refined cut-tree

Let r be a cut vertex in the refined cut-tree T_c^G rooted at cut vertex c of block graph G . Let T_r^G be the subtree of tree T_c^G rooted at cut vertex r . If R denotes the set containing the vertex r and all its descendants, then $T_r^G = T_c^G[R]$. Let G_r denote the subgraph of G reconstructed from the subtree T_r^G using following construction:

Construction 1: Let \mathcal{C}_i denotes the set of cut vertices of the tree T_r^G , and let \mathcal{B}_i denote the set of block vertices of the tree T_r^G . Then $V(G_r) = \mathcal{C}_i \cup \{B \mid B \in \mathcal{B}_i\}$, and $G_r = G[V(G_r)]$.

We compute the following parameters at every cut vertex node r of the tree T_c^G .

$A_r(G_r) = \text{Min} \{|D| : r \in D \text{ and } D \text{ is a restrained dominating set of } G_r\}$.

$B_r(G_r) = \text{Min} \{|D| : r \notin D \text{ and } D \text{ is a restrained dominating set of } G_r\}$.

$C_r(G_r) = \text{Min} \{|D| : r \notin D, D \text{ dominates } V(G_r) \setminus \{r\} \text{ and every vertex } v \notin D \text{ has an adjacent vertex } w \notin D\}$.

$D_r(G_r) = \text{Min} \{|D| : r \notin D, D \text{ dominates } V(G_r) \setminus \{r\} \text{ and every vertex } v \notin D \cup \{r\} \text{ has an adjacent vertex } w \notin D\}$.

$E_c(G_r) = \text{Min} \{|D| : r \notin D, D \text{ dominates } V(G_r) \text{ and every vertex } v \notin D \cup \{r\} \text{ has an adjacent vertex } w \notin D\}$.

5.3 Parameters to be computed at a block vertex in the refined cut-tree

Suppose T_c^G is the refined cut-tree rooted at the cut vertex c and B_i is a block vertex of T_c^G . Let $T_{B_i}^G$ denote the subtree of the refined cut-tree T_c^G rooted at the vertex B_i . Also suppose that BG_i denotes the graph reconstructed from the tree $T_{B_i}^G$ using Construction 1. Note that the tree $T_{B_i}^G$ is not necessarily the refined cut-tree of the graph BG_i . Let BC_i denote the block corresponding to the block vertex B_i . Now we compute the following parameters at every block vertex B_i of tree T_c^G .

$A_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i) \setminus V(BC_i) \text{ and every vertex } v \in V(BG_i) \setminus D \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$B_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i) \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$F_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ does not belong to } D, \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$H_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ belongs to } D, \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$I_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ does not belong to } D, \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ belongs to } D \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

If B_i is empty, then we also define the following three parameters:

$C_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ does not belong to } D, \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$D_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ belongs to } D, \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

$E_{B_i}(BG_i) = \text{Min } \{|D| : D \text{ dominates } V(BG_i), \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ does not belong to } D, \text{ at least one vertex of } V(BC_i) \cap V(BG_i) \text{ belongs to } D \text{ and every vertex } v \in V(BG_i) \setminus (D \cup V(BC_i)) \text{ has an adjacent vertex } w \in V(BG_i) \setminus D\}.$

5.4 Computation of the parameters at a leaf vertex in the refined cut-tree

Rule 1:

Let B_i be a leaf vertex of tree T_c^G , and BC_i be the corresponding block of G . Note that every leaf vertex of tree is a block vertex. In this case, $T_{B_i}^G$ consists of only one block-vertex B_i , and $BG_i = G[B_i]$. Note that $B_i \neq \emptyset$ and $V(BG_i) \setminus V(BC_i) = \emptyset$. Now, the parameters can be computed in the following way:

$$A_{B_i}(BG_i) = \begin{cases} 1, & \text{if } |B_i| = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$B_{B_i}(BG_i) = 1$$

.

$$F_{B_i}(BG_i) = \begin{cases} \infty, & \text{if } |B_i| = 1 \\ 1, & \text{otherwise} \end{cases}$$

$$H_{B_i}(BG_i) = 1$$

$$I_{B_i}(BG_i) = \begin{cases} \infty, & \text{if } |B_i| = 1 \\ 1, & \text{otherwise} \end{cases}$$

5.5 Computation of the parameters at a cut vertex in the refined cut-tree

Rule 2:

Let r be a cut vertex of the refined cut-tree T_c^G . Let T_r^G be the subtree of T_c^G rooted at r and G_r be the graph reconstructed from the tree T_r^G , as discussed earlier. Also suppose that the vertex r is having k children, say B_1, B_2, \dots, B_k , in the tree T_c^G . Let BC_1, BC_2, \dots, BC_k be the blocks of G corresponding to the block vertices B_1, B_2, \dots, B_k , respectively. Let $T_{B_1}^G, T_{B_2}^G, \dots, T_{B_k}^G$ be the subtrees of the tree T_c^G , and BG_1, BG_2, \dots, BG_k be the subgraphs of G reconstructed from the trees $T_{B_1}^G, T_{B_2}^G, \dots, T_{B_k}^G$, respectively. Then, the parameters corresponding to cut vertex r can be computed in the following way:

Parameter $A_r(G_r)$

Suppose D is a minimum cardinality restrained dominating set of G_r containing the vertex r . Then r dominates all the vertices of the blocks BC_1, BC_2, \dots, BC_k . Hence, the parameter $A_r(G_r)$ can be defined in the following way:

$$A_r(G_r) = 1 + \sum_{i=1}^k A_{B_i}(BG_i).$$

Parameter $B_r(G_r)$

Suppose D is a minimum cardinality restrained dominating set of G_r not containing the vertex r . Since B_1, B_2, \dots, B_k are children of r in the tree T_r^G , the cut vertex r belong to k blocks BC_1, BC_2, \dots, BC_k , where $k \geq 1$. Now we may have two possibilities for the set D .

- (i) There exists an index i , $1 \leq i \leq k$, such that $V(BG_i) \setminus D \neq \emptyset$, and $V(BG_i) \cap D \neq \emptyset$. In this case $|D| = \phi = \text{Min}_i(I_{B_i}(BG_i) + \sum_{j=1, (j \neq i)}^k B_{B_j}(BG_j))$.
- (ii) There exist indices i, j , $1 \leq i, j \leq k$, such that $V(BG_i) \setminus D \neq \emptyset$, and $V(BG_j) \cap D \neq \emptyset$. In this case $|D| = \psi = \text{Min}_{i,j (i \neq j)}(F_{B_i}(BG_i) + H_{B_j}(BG_j) + \sum_{m=1, (m \neq i,j)}^k B_{B_m}(BG_m))$.

$$\text{Hence } B_r(G_r) = \text{Min}(\phi, \psi).$$

Parameter $C_r(G_r)$

$C_r(G_r)$ can be computed as follows.

$$C_r(G_r) = \text{Min}_i(F_{B_i}(BG_i) + \sum_{j=1, (j \neq i)}^k B_{B_j}(BG_j)).$$

Parameter $D_r(G_r)$

Let D be a minimum cardinality set not containing r such that D dominates all the vertices of G_r except r , and every vertex $v \notin D \cup \{r\}$ has an adjacent vertex $w \notin D$. Now, each vertex in the block $V(BC_i)$, for all i , $1 \leq i \leq k$, is adjacent to the vertex r , which is not in D . Hence $D_r(G_r)$ is equal to the cardinality of a minimum cardinality set D not containing r such that D dominates all the vertices of G_r except r , and every vertex $v \notin D \cup (\cup_{i=1}^k BC_i)$ has an adjacent vertex $w \notin D$. So, the parameter $D_r(G_r)$ can be defined in the following way:

$$D_r(G_r) = \sum_{i=1}^k B_{B_i}(BG_i).$$

Parameter $E_r(G_r)$

Here we have two cases to consider.

If at least one of the B_i is non-empty, then

$$E_r(G_r) = \sum_{i=1}^k B_{B_i}(BG_i),$$

otherwise, if all the B_i 's are empty, then

$$E_r(G_r) = \text{Min}_i(D_{B_i}(BG_i) + \sum_{j=1, (j \neq i)}^k B_{B_j}(BG_j)).$$

5.6 Computation of the parameters at a non-leaf block vertex in the refined cut-tree

Rule 3:

Let B_k be a non-leaf block vertex of tree T_c^G and BC_k be the corresponding block of G . Let $T_{B_k}^G$ denote the subtree of the tree T_c^G rooted at the vertex B_k . Let BG_k denote the graph reconstructed from the tree $T_{B_k}^G$ as discussed earlier. Also suppose that x_1, x_2, \dots, x_p are children of B_k in the tree T_c^G . Let $T_{x_1}^G, T_{x_2}^G, \dots, T_{x_p}^G$ be the subtrees of the tree T_c^G rooted at the vertices x_1, x_2, \dots, x_p respectively, and $G_{x_1}, G_{x_2}, \dots, G_{x_p}$ be subgraphs of G reconstructed from the trees $T_{x_1}^G, T_{x_2}^G, \dots, T_{x_p}^G$, respectively. Then, the parameters corresponding to block vertex B_k can be computed in the following way:

Parameter $A_{B_k}(BG_k)$

Let D be a minimum cardinality subset of $V(BG_i)$ such that D dominates $V(BG_i) \setminus V(BC_i)$, and every vertex $v \in V(BG_i) \setminus D$ has an adjacent vertex $w \in V(BG_i) \setminus D$. Here we have three cases to consider depending on the cardinality of the set B_k .

Case 1: $|B_k| \geq 2$.

In this case, we have $A_{B_k}(BG_k) = \sum_{i=1}^p \text{Min}(A_{x_i}(G_{x_i}), D_{x_i}(G_{x_i}))$.

Case 2: $|B_k| = 1$.

Again we have following subcases to consider.

Subcase 2.1 : $|B_k| = 1$ and the vertex of B_k does not belong to D .

Here $|D| = D1 = \text{Min}_i(D_{x_i}(G_{x_i}) + \sum_{j=1, (j \neq i)}^p \text{Min}(A_{x_j}(G_{x_j}), D_{x_j}(G_{x_j})))$.

Subcase 2.2 : $|B_k| = 1$ and the vertex of B_k belongs to D .

This is possible only if all the children of B_k belong to D , that is, $\{x_1, x_2, \dots, x_p\} \subseteq D$.

Here $|D| = D2 = 1 + \sum_{i=1}^p A_{x_i}(G_{x_i})$.

Hence, in this case $A_{B_k}(BG_k) = \text{Min}(D1, D2)$.

Case 3: $|B_k| = 0$.

Again we have following subcases to consider.

Subcase 3.1 : B_k has exactly one child, say, x_1 .

$A_{B_k}(BG_k) = \text{Min}(A_{x_1}(G_{x_1}), C_{x_1}(G_{x_1}))$.

Subcase 3.2 : B_k has two or more child. Again there are three possibilities:

(i) All the children of B_k in the tree $T_{B_k}^G$ belong to D .

Here $|D| = D1 = \sum_{i=1}^p A_{x_i}(G_{x_i})$.

(ii) Exactly one child of B_k does not belong to D .

Here $|D| = D2 = \text{Min}_i(C_{x_i}(G_{x_i}) + \sum_{j=1, (j \neq i)}^p A_{x_j}(G_{x_j}))$.

(iii) At least two child of B_k do not belong to D .

Here $|D| = D3 = \text{Min}_{i,j(i \neq j)}(D_{x_i}(G_{x_i}) + D_{x_j}(G_{x_j}) + \sum_{k=1, (k \neq i,j)}^p \text{Min}(A_{x_k}(G_{x_k}), D_{x_k}(G_{x_k})))$.
Hence, in this subcase $A_{B_k}(BG_k) = \text{Min}(D1, D2, D3)$.

Parameter $B_{B_k}(BG_k)$

Let D be the minimum cardinality subset of the set $V(BG_k)$ such that D dominates $V(BG_k)$, and every vertex $v \in V(BG_k) \setminus (D \cup BC_k)$ has an adjacent vertex $w \in V(BG_k) \setminus D$. Here we have two cases to consider depending on the cardinality of the set B_k .

Case 1: B_k is non-empty.

Again there are two possibilities:

(i) A vertex $x \in B_k$ belongs to D .

Here $|D| = D1 = 1 + \sum_{i=1}^p \text{Min}(A_{x_i}(G_{x_i}), D_{x_i}(G_{x_i}))$.

(ii) No vertex of B_k belongs to D .

Here $|D| = D2 = \text{Min}_i(A_{x_i}(G_{x_i}) + \sum_{j=1, (j \neq i)}^p \text{Min}(A_{x_j}(G_{x_j}), D_{x_j}(G_{x_j})))$.

Hence, in this case $B_{B_k}(BG_k) = \text{Min}(D1, D2)$.

Case 2: B_k is empty.

Again there are two possibilities to consider.

(i) No child of B_k belongs to D (that is, $\{x_1, x_2, \dots, x_p\} \cap D = \emptyset$).

Here $|D| = D1 = \sum_{i=1}^p E_{x_i}(G_{x_i})$.

(ii) At least one child of B_k belongs to D (that is, $\{x_1, x_2, \dots, x_p\} \cap D \neq \emptyset$).

Here $|D| = D2 = \text{Min}_i(A_{x_i}(G_{x_i}) + \sum_{j=1, (j \neq i)}^p \text{Min}(A_{x_j}(G_{x_j}), D_{x_j}(G_{x_j})))$.

Hence, in this case $B_{B_k}(BG_k) = \text{Min}(D1, D2)$.

Parameter $E_{B_k}(BG_k)$

Note that this parameter is computed only if B_k is an empty set.

If B_k has exactly one child in the tree $T_{B_k}^G$, that is, $p = 1$, then $E_{B_k}(BG_k) = \infty$.

Otherwise $E_{B_k}(BG_k) = \text{Min}_{i,j}(A_{x_i}(G_{x_i}) + D_{x_j}(G_{x_j}) + \sum_{m=1, (m \neq i,j)}^p \text{Min}(A_{x_m}(G_{x_m}), D_{x_m}(G_{x_m})))$.

Parameter $D_{B_k}(BG_k)$

Note that this parameter is computed only if B_k is an empty set.

$D_{B_k}(BG_k) = \text{Min}_i(A_{x_i}(G_{x_i}) + \sum_{j=1, (j \neq i)}^p \text{Min}(A_{x_j}(G_{x_j}), D_{x_j}(G_{x_j})))$.

Parameter $C_{B_k}(BG_k)$

Note that this parameter is computed only if B_k is an empty set. Let D be a minimum cardinality subset of $V(BG_k)$ such that D dominates $V(BG_k)$, at least one vertex of $BC_k \cap V(BG_k)$ does not belong to D and every vertex $v \in V(BG_k) \setminus (D \cup BC_k)$ has an adjacent vertex $w \in V(BG_k) \setminus D$. Here we have two cases to consider.

Case 1: No child of B_k belong to D .

Here $|D| = D1 = \sum_{i=1}^p E_{x_i}(G_{x_i})$.

Case 2: At least one child of B_k belongs to D .

Here $|D| = D2 = E_{B_k}(BG_k)$.

Hence $C_{B_k}(BG_k) = \text{Min}(D1, D2)$.

Parameter $F_{B_k}(BG_k)$

Since B_k is not a leaf vertex of tree T_c^G , $|B_k| \neq |BG_k|$. Now we have two cases to consider depending on the cardinality of B_k .

Case 1: B_k is non-empty.

Here $F_{B_k}(BG_k) = B_{B_k}(BG_k)$.

Case 2: B_k is empty.

Here $F_{B_k}(BG_k) = C_{B_k}(BG_k)$.

Parameter $H_{B_k}(BG_k)$

Since B_k is not a leaf vertex of tree T_c^G , $|B_k| \neq |BG_k|$. Now we have two cases to consider depending on the cardinality of B_k .

Case 1: B_k is non-empty.

$H_{B_k}(BG_k) = B_{B_k}(BG_k)$.

Case 2: B_k is empty.

$I_{B_k}(BG_k) = D_{B_k}(BG_k)$.

Parameter $I_{B_k}(BG_k)$

We have two cases to consider depending on the cardinality of B_k .

Case 1: B_k is non-empty.

Here $I_{B_k}(BG_k) = B_{B_k}(BG_k)$.

Case 2: B_k is empty.

Here $I_{B_k}(BG_k) = E_{B_k}(BG_k)$.

5.7 Algorithm

We are now ready to propose an algorithm to compute $\gamma_r(G)$ of a block graph $G = (V, E)$ having at least two blocks.

Algorithm 1 Algorithm-RD(G)

Input: A block graph $G = (V, E)$ with at least two blocks.

Output: $\gamma_r(G)$.

begin

 Compute the refined cut-tree T of G rooted at a cut vertex c of G ;

 Compute a reverse BFS ordering $\alpha = (v_1, v_2, \dots, v_n = c)$ of T ;

for $i=1$ **to** n **do**

if v_i *is a leaf vertex of* T **then**

 Compute the parameters at v_i using Rule 1;

else if v_i *is a cut vertex of* T **then**

 Compute the parameters at v_i using Rule 2;

else if v_i *is a non-leaf block vertex of* T **then**

 Compute the parameters at v_i using Rule 3;

$\gamma_r(G) = \text{Min}(A_c(G), B_c(G))$ for the root c of T .

return $\gamma_r(G)$;

5.8 Illustration of the algorithm

We now illustrate our algorithm using an example. Fig. 2 shows an example of block graph G having seven blocks, $BC_1 = G[\{v_5, v_6, v_7\}]$, $BC_2 = G[\{v_2, v_3, v_4, v_5\}]$, $BC_3 = G[\{v_1, v_2\}]$, $BC_4 = G[\{v_7, v_8, v_9, v_{10}\}]$, $BC_5 = G[\{v_{10}, v_{11}\}]$, $BC_6 = G[\{v_{11}, v_{12}\}]$, $BC_7 = G[\{v_9, v_{13}, v_{14}\}]$ and the corresponding refined cut-tree rooted at vertex v_6 , say $T_{v_6}^G$. The cut-tree $T_{v_6}^G$ is having 13 vertices, in which six are cut vertices $v_2, v_5, v_7, v_9, v_{10}, v_{11}$ and seven are block vertices $B_1 = \{v_6\}$, $B_2 = \{v_3, v_4\}$, $B_3 = \{v_1\}$, $B_4 = \{v_8\}$, $B_5 = \emptyset$, $B_6 = \{v_{12}\}$, $B_7 = \{v_{13}, v_{14}\}$.

For the refined cut-tree of G shown in Fig 2, the reverse of BFS ordering is:

$$\alpha = (B_6, B_3, v_{11}, v_2, B_7, B_5, B_2, v_9, v_{10}, v_5, B_4, B_1, v_7).$$

- (1) B_6 is a block vertex, BC_6 is an end block of G , $|B_6| = 1$, $|V(BC_6)| = 2$.
 $A_{B_6} = 1$, $B_{B_6} = 1$, $F_{B_6} = \infty$, $H_{B_6} = 1$, $I_{B_6} = \infty$.
- (2) B_3 is a block vertex, BC_3 is an end block of G , $|B_3| = 1$, $|V(BC_3)| = 2$.
 $A_{B_3} = 1$, $B_{B_3} = 1$, $F_{B_3} = \infty$, $H_{B_3} = 1$, $I_{B_3} = \infty$.
- (3) v_{11} is a cut vertex having only one child B_6 in the tree $T_{v_6}^G$ and $|B_6| = 1$.
 $A_{v_{11}} = 1 + A_{B_6} = 2$, $B_{v_{11}} = I_{B_6} = \infty$, $C_{v_{11}} = F_{B_6} = \infty$, $D_{v_{11}} = B_{B_6} = 1$, $E_{v_{11}} = B_{B_6} = 1$.
- (4) v_2 is a cut vertex having only one child B_3 in the tree $T_{v_6}^G$ and $|B_3| = 1$.
 $A_{v_2} = 1 + A_{B_3} = 2$, $B_{v_2} = I_{B_3} = \infty$, $C_{v_2} = F_{B_3} = \infty$, $D_{v_2} = B_{B_3} = 1$, $E_{v_2} = B_{B_3} = 1$.
- (5) B_7 is a block vertex, BC_7 is an end block of G , $|B_7| = 2$, $|V(BC_7)| = 3$.
 $A_{B_7} = 0$, $B_{B_7} = 1$, $F_{B_7} = 1$, $H_{B_7} = 1$, $I_{B_7} = 1$.
- (6) B_5 is a block vertex, BC_5 is a block of G , $|B_5| = 0$, $|V(BC_5)| = 2$, and B_5 is having one child v_{11} in the tree $T_{v_6}^G$.
 $A_{B_5} = \min(A_{v_{11}}, C_{v_{11}}) = 2$, $B_{B_5} = \min(E_{v_{11}}, A_{v_{11}}) = 1$, $E_{B_5} = \infty$, $D_{B_5} = A_{v_{11}} = 2$,
 $C_{B_5} = \min(E_{v_{11}}, E_{B_5}) = 1$, $F_{B_5} = C_{B_5} = 1$, $H_{B_5} = D_{B_5} = 2$, $I_{B_5} = E_{B_5} = \infty$.
- (7) B_2 is a block vertex, BC_2 is a block of G , $|B_2| = 2$, $|V(BC_2)| = 4$, and B_2 is having one child v_1 in the tree $T_{v_6}^G$.
 $A_{B_2} = \min(B_{v_2}, 1 + A_{v_2}) = 1$, $B_{B_2} = \min(1 + \min(A_{v_2}, B_{v_2}), A_{v_2}) = 2$, $F_{B_2} = 2$, $H_{B_2} = 2$,
 $I_{B_2} = 2$.
- (8) v_9 is a cut vertex having only one child B_7 in the tree $T_{v_6}^G$ and $|B_7| = 2$.
 $A_{v_9} = 1 + A_{B_7} = 1$, $B_{v_9} = I_{B_7} = 1$, $C_{v_9} = F_{B_7} = 1$, $D_{v_9} = B_{B_7} = 1$, $E_{v_9} = B_{B_7} = 1$.
- (9) v_{10} is a cut vertex having only one child B_5 in the tree $T_{v_6}^G$ and $|B_5| = 0$.
 $A_{v_{10}} = 1 + A_{B_5} = 3$, $B_{v_{10}} = I_{B_5} = \infty$, $C_{v_{10}} = F_{B_5} = 1$, $D_{v_{10}} = B_{B_5} = 1$, $E_{v_{10}} = D_{B_5} = 2$.
- (10) v_5 is a cut vertex having only one child B_2 in the tree $T_{v_6}^G$ and $|B_2| = 2$.
 $A_{v_5} = 1 + A_{B_2} = 2$, $B_{v_5} = I_{B_2} = 2$, $C_{v_5} = F_{B_2} = 2$, $D_{v_5} = B_{B_2} = 2$, $E_{v_5} = B_{B_2} = 2$.
- (11) B_4 is a block vertex, BC_4 is a block of G , $|B_4| = 1$, $|V(BC_4)| = 4$, and B_4 is having two child v_{10} and v_9 in the tree $T_{v_6}^G$.
 $A_{B_4} = \min(D_1, D_2)$ where $D_1 = \min(B_{v_{10}} + \min(A_{v_9}, B_{v_9}), B_{v_9} + \min(A_{v_{10}}, B_{v_{10}})) = 2$
and $D_2 = 1 + A_{v_9} + A_{v_{10}} = 5$, So $A_{B_4} = 2$.
 $B_{B_4} = \min(D_1, D_2)$ where $D_1 = 1 + \min(A_{v_9}, B_{v_9}) + \min(A_{v_{10}}, B_{v_{10}}) = 3$ and $D_2 =$

$Min(A_{v_9} + Min(A_{v_{10}}, B_{v_{10}}), A_{v_{10}} + Min(A_{v_9}, B_{v_9})) = 2$, So $B_{B_4} = 2$.
 $F_{B_4} = 2, H_{B_4} = 2, I_{B_4} = 2$.

- (12) B_1 is a block vertex, BC_1 is a block of G , $|B_1| = 1$, $|V(BC_1)| = 3$, and B_1 is having one child v_5 in the tree $T_{v_6}^G$.
 $A_{B_1} = Min(B_{v_5}, 1 + A_{v_5}) = 2$, $B_{B_1} = Min(1 + Min(A_{v_5}, B_{v_5}), A_{v_5}) = 2$, $F_{B_1} = 2$, $H_{B_1} = 2$,
 $I_{B_1} = 2$.
- (13) v_7 is a cut vertex having two child B_1 and B_4 in the tree $T_{v_6}^G$ and $|B_1| = |B_4| = 1$.
 $A_{v_7} = 1 + A_{B_1} + A_{B_4} = 5$, $B_{v_7} = Min(I_{B_1} + B_{B_4}, I_{B_4} + B_{B_1}, F_{B_1} + H_{B_4}, F_{B_4} + H_{B_1}) = 4$,
 $C_{v_7} = Min(F_{B_1} + B_{B_4}, F_{B_4} + B_{B_1}) = 4$, $D_{v_7} = B_{B_1} + B_{B_4} = 4$, $E_{v_7} = B_{B_1} + B_{B_4} = 4$.

Thus the minimum restrained domination number $\gamma_r(G) = Min(A_{v_7}, D_{v_7}) = 4$.

5.9 Complexity details

Theorem 5.1. *The restrained domination number of a block graph G can be computed in $O(n^4)$ time, where n denotes the number of vertices in G .*

Proof. The proof of correctness of Algorithm_RD follows from the recurrence relations obtained in the earlier part of this section.

Next we analyze the running time of the algorithm.

Let $G = (V, E)$ be a block graph with n vertices and m edges. The refined cut-tree T of G can be computed in $O(n+m)$ time using depth-first search similar to the use of depth-first search in constructing a cut-tree of G . Note that the number of vertices in the tree T is also $O(n)$. While constructing the tree T , we can also maintain the following information for each vertex v of the refined cut-tree: (i) v is a cut vertex or block vertex, (ii) if v is a block vertex, then the number of vertices in the corresponding block, and (iii) if v is a block vertex, then number of non-cut vertices in the corresponding block. All these information can be maintained in linear time.

Next we find a reverse of BFS ordering, say α , of the vertices of the refined cut-tree T . The ordering α can be computed in $O(n)$ time. Now we process the vertices of T in the ordering α , and for each vertex we compute some parameters using Rules 1, 2, and 3, depending on the case whether it is a leaf vertex (block vertex), non-leaf block vertex, or a cut vertex. If v is a block vertex and leaf of tree T , then we process it according to Rule 1, and all the parameters can be computed in $O(1)$ time. If v is a non-leaf block vertex, we process it according to Rule 3. If v is a cut vertex, we process it according to Rule 2.

Let $T_1(n)$ be the time required to compute all the parameters for any cut vertex given all the computed information for its children. Similarly, let $T_2(n)$ be the time required to compute all the parameters for any non-leaf block vertex given all the computed information for its children. Then the time complexity of the algorithm is $T(n) = O(n + m) + O(nT_1(n) + nT_2(n))$. Now we only need to find $T_1(n)$ and $T_2(n)$.

Let $c(v)$ denote the number of children of v in tree T , then $d_T(v) = c(v) + 1$.

Estimate for $T_1(n)$:

From the formula for A_v, B_v, C_v, D_v , and E_v , it is easy to note that A_v can be computed in $O(c(v)) = O(d_T(v))$ time, B_v can also be computed in $O(d_T(v))$ time, C_v can be computed in $O(d_T(v)^2)$ time, D_v can be computed in $O(d_T(v)^3)$ time, and E_v can be computed in $O(d_T(v)^2)$ time. Hence $T_1(n) = O(d_T(v)^3) = O(n^3)$.

Estimate for $T_2(n)$:

It is easy to note from the formula for $A_B, B_B, C_B, D_B, E_B, F_B, H_B$, and I_B , that A_B can be computed

in $O(d_T(B)^3)$ time, B_B can be computed in $O(d_T(B)^2)$ time, E_B can be computed in $O(d_T(B)^3)$ time, D_B can be computed in $O(d_T(B)^2)$ time, C_B can be computed in $O(d_T(B))$ time, F_B can be computed in $O(1)$ time, H_B can be computed in $O(1)$ time, and I_B can be computed in $O(1)$ time. Hence $T_2(n) = O(d_T(B)^3) = O(n^3)$.

Therefore $T_1(n) + T_2(n) = O(n^3)$, and hence $T(n) = O(n^4)$.

Hence our algorithm can be implemented in $O(n^4)$ time and the correctness of the algorithm follows from the recurrences defined in above subsections. Therefore, the theorem is true. \square

Note that a minimum cardinality restrained dominating set of a block graph G can be computed in $O(n^4)$ time by maintaining the sets corresponding to the computed parameters at every node of the refined cut-tree T of G .

6 Restrained domination in threshold graphs

In this section we give a method to compute a minimum restrained dominating set of a threshold graph G in linear-time. A graph $G = (V, E)$ is called a *threshold graph* if there is a real number T and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(v) \leq T$ [4]. Many characterizations of threshold graphs are available in the literature. An important characterization of threshold graph, which is used in designing polynomial time algorithms is following:

A graph G is threshold graph if and only if it is a split graph and, for any split partition (C, I) of G , there is an ordering (x_1, x_2, \dots, x_p) of the vertices of C such that $N_G[x_1] \subseteq N_G[x_2] \subseteq \dots \subseteq N_G[x_p]$, and there is an ordering (y_1, y_2, \dots, y_q) of the vertices of I such that $N_G(y_1) \supseteq N_G(y_2) \supseteq \dots \supseteq N_G(y_q)$ [22].

Theorem 6.1. *Let $G = (V, E)$ be a threshold graph having at least three vertices with split partition (C, I) as defined above, then a minimum restrained dominating set D_r^* of G can be computed in the following way:*

$$D_r^* = \begin{cases} V, & \text{if } p = 1 \\ \{x_p\}, & \text{if } p > 1 \text{ and } N_G[x_p] = N_G[x_{p-1}] \\ \{x_p\} \cup \{v \in I \mid v \in N_G(x_p) \setminus N_G(x_{p-1})\}, & \text{if } p > 1 \text{ and } N_G[x_p] \neq N_G[x_{p-1}] \end{cases}$$

Proof. We know that a restrained dominating set must contain all the pendant vertices of graph. If $|C| = 1$, then the only non-pendant vertex is the vertex in the partite set C , say v_c . Then I must be contained in D^* , where D^* is any restrained dominating set of G . Since v_c is an isolated vertex in $G[V \setminus I]$, by the definition of restrained dominating set, the vertex v_c should also belongs to D^* . Thus $V = \{v_c\} \cup I$ is the only restrained dominating set of G in the case when $|C| = 1$. Hence $D^* = V$.

If $|C| > 1$ and $N_G[x_p] = N_G[x_{p-1}]$, then $D^* = \{x_p\}$ is a dominating set of G . Note that D^* is also a restrained dominating set of G since every vertex in $V \setminus D^*$ except v_{p-1} is adjacent to the vertex v_{p-1} . Also $|D^*| = 1$, and hence $D^* = \{x_p\}$ is a minimum restrained dominating set of G .

Now consider the case when $|C| > 1$ and $N_G[x_p] \neq N_G[x_{p-1}]$. Let D_r^* be a minimum restrained dominating set of G . The set $S = N_G(x_p) \setminus N_G(x_{p-1})$ can be dominated by the vertex x_p or the set S itself. Hence either $x_p \in D_r^*$ or $S \subseteq D_r^*$.

Since all the vertices of S are isolated in $G[V \setminus \{x_p\}]$, if $x_p \in D_r^*$, then S must be contained in D_r^* . Hence $\{x_p\} \cup S \subseteq D_r^*$. But since $\{x_p\} \cup S$ is a restrained dominating set of G , $D_r^* = \{x_p\} \cup S$.

If $S \subseteq D_r^*$, then since S does not dominate the vertex x_{p-1} , $|D_r^*| \geq |S| + 1$. Hence $\{x_p\} \cup S$ is a minimum restrained dominating set of G . This completes the proof of the theorem. \square

Given a split partition (C, I) of threshold graph G and an ordering (x_1, x_2, \dots, x_p) of the vertices of C such that $N_G[x_1] \subseteq N_G[x_2] \subseteq \dots \subseteq N_G[x_p]$, and an ordering (y_1, y_2, \dots, y_q) of the vertices of I such that $N_G(y_1) \supseteq N_G(y_2) \supseteq \dots \supseteq N_G(y_q)$, one can compute in $O(n + m)$ time a minimum cardinality restrained dominating set of a threshold graphs.

7 Restrained domination in cographs

In this section, we show that the MINIMUM RESTRAINED DOMINATION problem can also be solved in linear time for cographs, which is a super class of threshold graphs. A *cograph* is a graph without induced P_4 [5]. Various characterizations of cographs are known in literature.

Theorem 7.1. *A graph is a cograph if and only if every induced subgraph H is disconnected or the complement \overline{H} is disconnected.*

A cograph has a tree decomposition which is called a cotree. A cotree is a pair (T, f) comprising a rooted binary tree T together with a bijection f from the vertices of the graph to the leaves of the tree. Each internal node of T has a label \otimes or \oplus . The operator \otimes is called a join operation and it makes every vertex that is mapped to a leaf in the left subtree adjacent to every vertex that is mapped to leaf in the right subtree. The operator \oplus is called union operation. In that case the graph is the union of the graphs defined by the left and right subtree. If n denotes the number of vertices in the graph, its cotree has $O(n)$ nodes. A cotree decomposition of a graph can also be obtained in linear-time [6]. Suppose $I(G)$ denotes the number of isolated vertices in a graph G . The following result regarding the domination number of a cograph is already proved.

Theorem 7.2. [20] *If G is a cograph with at least two vertices, then*

$$\gamma(G) = \begin{cases} \gamma(G_1) + \gamma(G_2), & \text{if } G = G_1 \oplus G_2 \\ \min\{\gamma(G_1), \gamma(G_2), 2\}, & \text{if } G = G_1 \otimes G_2 \end{cases}$$

Now we are ready to prove the following theorem.

Theorem 7.3. *If G is a cograph with at least two vertices, then*

$$\gamma_r(G) = \begin{cases} \gamma_r(G_1) + \gamma_r(G_2), & \text{if } G = G_1 \oplus G_2 \\ 2, & \text{if } G = G_1 \otimes G_2, |V(G_1)| = 1, |V(G_2)| = 1 \\ \min\{\gamma(G_1), \gamma(G_2), 2\}, & \text{if } G = G_1 \otimes G_2, |V(G_1)| \geq 2, |V(G_2)| \geq 2 \\ \min\{1 + I(G_2), \gamma(G_2)\} & \text{if } G = G_1 \otimes G_2, |V(G_1)| = 1, |V(G_2)| \geq 2 \\ \min\{1 + I(G_1), \gamma(G_1)\} & \text{if } G = G_1 \otimes G_2, |V(G_1)| \geq 2, |V(G_2)| = 1 \end{cases}$$

Proof. For the case $G = G_1 \oplus G_2$, clearly $\gamma_r(G) = \gamma_r(G_1) + \gamma_r(G_2)$, since no vertex of G_1 is adjacent to any vertex of G_2 .

When $G = G_1 \otimes G_2$ and $|V(G_1)| = |V(G_2)| = 1$, then $G = K_2$ and hence $\gamma_r(G) = 2$.

Now consider the case when $G = G_1 \otimes G_2$, $|V(G_1)| \geq 2$, $|V(G_2)| \geq 2$. Here $\{x, y\}$ is a restrained dominating set of G where $x \in G_1$, $y \in G_2$. If there exists a vertex $x \in G_1$ which dominates all the vertices of G_1 , then $\{x\}$ is a restrained dominating set of G_1 . Similarly if there exists a vertex $y \in G_2$ which dominates all the vertices of G_2 , then $\{y\}$ is a restrained dominating set of G_2 . This proves the formula in this case.

Now consider the case when $G = G_1 \otimes G_2$, $|V(G_1)| = 1$, $|V(G_2)| \geq 2$. Let $V(G_1) = \{v\}$. Suppose D_r^* is a minimum restrained dominating set of G . If $v \in D_r^*$, then $D_r^* = \{v\} \cup I(G_2)$. If $v \notin D_r^*$, then D_r^* is a minimum dominating set of G_2 . This proves the formula in this case.

The formula for the case $G = G_1 \otimes G_2$, $|V(G_1)| \geq 2$, $|V(G_2)| = 1$ can be proved similarly. \square

Using the above theorem, a minimum restrained dominating set of a co-graph can be computed in $O(n + m)$ time.

8 Restrained domination in chain graphs

A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a chain, that is, the vertices of X can be linearly ordered, say x_1, x_2, \dots, x_p , such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \dots \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain [25]. An ordering $\alpha = (x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q)$ of $X \cup Y$ is called a chain ordering if $N_G(x_1) \subseteq N_G(x_2) \subseteq \dots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \dots \supseteq N_G(y_q)$. It is well known that every chain graph admits a chain ordering [25, 21].

Lemma 8.1. *Let G be a connected chain graph as defined above and v be a pendant vertex of G . If $v \in Y$, then v is adjacent to x_p and if $v \in X$, then v is adjacent to y_1 .*

Proof. By the definition of chain ordering, every vertex y in Y is adjacent to x_p , similarly every vertex x in X is adjacent to y_1 . Hence the lemma is proved. \square

Lemma 8.2. *Let G be a chain graph with at least three vertices. If every non-pendant vertex of G is adjacent to a pendant vertex as well as a non-pendant vertex, then G is a bi-star.*

Proof. If a vertex v in G is adjacent to a pendant vertex then either $v = x_p$ or $v = y_1$. Since every non-pendant vertex of G is adjacent to a pendant vertex, G has at most two non-pendant vertices. To show that G is a bi-star, we need to show that G has exactly two non-pendant vertices. Clearly G has at least one non-pendant vertex, say v . By the statement of lemma, v must have an adjacent non-pendant vertex. Therefore G contains exactly two non-pendant vertices. \square

Theorem 8.1. *Let $G = (X, Y, E)$ be a connected chain graph having at least three vertices and $\alpha = (x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q)$ is chain ordering of $X \cup Y$. Then $t \leq \gamma_c(G) \leq t + 2$, where t denotes the number of pendant vertices of G . Furthermore, the following are true.*

- (a) $\gamma_r(G) = t$ if and only if $G = K_2$ or bi-star.
- (b) Let P denotes the set of all pendant vertices of G and P_A denotes the set of vertices adjacent to the vertices of P . Then $\gamma_r(G) = t + 1$ if and only if either G or $G' = G[(X \cup Y) \setminus (P \cup P_A)]$ is a star.
- (c) If G is a graph other than the graphs described in the above statements then $\gamma_r(G) = t + 2$.

Proof. Since every restrained dominating set contains all the pendant vertices of graph, $\gamma_r(G) \geq t$. Now suppose that P denotes the set of all pendant vertices of graph G . Then the set $D = P \cup \{x_p, y_1\}$ is a dominating set of G . We claim that D is also a restrained dominating set of G . If not, then there exists a vertex $v \in (X \cup Y) \setminus D$ such that $N_G(v) \subseteq D$. Since $|N_G(v)| \geq 2$, at least one neighbor of v is a pendant vertex. But then by Lemma 8.1, v is either x_p or y_1 , which is a contradiction. Hence $D = P \cup \{x_p, y_1\}$ is a restrained dominating set of G . This proves that $\gamma_r(G) \leq t + 2$.

(a) If $G = K_2$ or bi-star, then $D = P$ is a restrained dominating set of G . Hence $\gamma_r(G) = t$.

Conversely suppose that $\gamma_r(G) = t$. Let D be the minimum restrained dominating set of G . Then D is exactly the set of all pendant vertices of G . Then we have two possibilities for graph G .

(i) All the vertices of graph G are pendant vertices. Then $G = K_2$.

(ii) Every non-pendant vertex is adjacent to a pendant vertex as well as a non-pendant vertex. By Lemma 8.2, chain graph satisfying this property is bi-star.

(b) If G is star, then clearly $\gamma_r(G) = |X \cup Y| = t + 1$. If G' is star with star center v , then $D = P \cup \{v\}$ is a dominating set of G . Also $x_p, y_1 \notin D$, and every vertex u not in D is either adjacent to x_p or y_1 . Hence D is also a restrained dominating set of G and $\gamma_r(G) = t + 1$.

Conversely suppose that $\gamma_r(G) = t + 1$. Let D be the minimum restrained dominating set of G . Then $P \subseteq D$. Now we have two possibilities for graph G .

(i) The set P dominates $X \cup Y$. But there exists a vertex $v \in (X \cup Y) \setminus P$ such that $N_G(v) \subseteq D$. Thus v is a non-pendant vertex in the graph G and all the neighbors of V are pendant. This is possible only when v is the only non-pendant vertex in the graph G . Hence G is a star in this case.

(ii) The set P does not dominate $X \cup Y$, but there exist a non-pendant vertex v such that $P \cup \{v\}$ dominates all the vertices of the graph. Hence v dominates the set $(X \cup Y) \setminus (P \cup P_A)$. This is possible only when $G[(X \cup Y) \setminus (P \cup P_A)]$ is a star.

(c) Proof directly follows from above statements. □

Now we are ready to prove the following theorem:

Theorem 8.2. *A minimum restrained dominating set of a chain graph can be computed in $O(n + m)$ time.*

Proof. For a chain graph $G = (X, Y, E)$, a chain ordering $\alpha = (x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q)$ of $X \cup Y$ can be computed in linear time. The set P of all pendant vertices of G can also be computed in linear time. Now, if $|X \cup Y| = 2$, then take $D = X \cup Y$. It can also be computed in linear time whether G is a star or bistar. If G is a bistar, then take $D = P$. If G is a star with star center v , then take $D = P \cup \{v\}$. Let S denote the set of vertices adjacent to a pendant vertex of G . Then the set S can also be computed in linear time. If $G' = G[(X \cup Y) \setminus (P \cup S)]$ is a star with star center u , then $D = P \cup \{u\}$. Otherwise, take $D = P \cup \{x_p, y_1\}$. By Theorem 8.1, D is minimum cardinality restrained dominating set of G . Hence, the theorem is proved. □

9 An upper bound for the restrained domination number

Zverovich and Pohosyan [27] proved the following result.

Theorem 9.1. *If a graph G with n vertices and minimum degree δ has a perfect matching, then*

$$\gamma_r(G) \leq \frac{2(1 + \ln(\delta + 1))}{\delta + 1}n + \epsilon$$

where $\epsilon = 0$ if n is even and $\epsilon = 1$ otherwise.

In this section we prove the following stronger result for the restrained domination number of a graph G .

Theorem 9.2. *Let G be a connected graph with n vertices and minimum degree δ , then G has a restrained dominating set of cardinality at most $\frac{2(1 + \ln(\delta + 1))}{\delta + 1}n$.*

Proof. Let $p \in [0, 1]$ be arbitrary. We pick randomly and independently each vertex of graph G with probability p . Let A be the set of all picked vertices, B_A be the random set of all vertices in $V \setminus A$ that do not have any neighbor in A , and C_A be the set of all vertices in $V \setminus (A \cup B_A)$ for which all the neighbors are in $A \cup B_A$. Then $E(|A|) = np$.

For each fixed vertex $v \in V$, $Pr(v \in B_A) = Pr(v \text{ and all its neighbors are not in } A) \leq (1 - p)^{\delta+1}$. Hence $E(|B_A|) \leq n(1 - p)^{\delta+1}$.

$$Pr(v \in A \cup B_A) = Pr(v \in A) + Pr(v \in B_A) \leq p + (1 - p)^{\delta+1}$$

$$Pr(v \notin A \cup B_A) = 1 - Pr(v \in A \cup B_A) = 1 - (Pr(v \in A) + Pr(v \in B_A)) \leq 1 - p$$

For each fixed vertex $v \in V$, $Pr(v \in C) = Pr(v \notin A \cup B_A \text{ and all its neighbors are in } A \cup B_A) \leq (1 - p)(p + (1 - p)^{\delta+1})^\delta$. Hence $E(|C_A|) \leq n(1 - p)(p + (1 - p)^{\delta+1})^\delta$.

Now

$$\begin{aligned} E(|A| + |B_A| + |C_A|) &\leq np + n(1 - p)^{\delta+1} + n(1 - p)(p + (1 - p)^{\delta+1})^\delta \\ &\leq np + n(1 - p)^{\delta+1} + n(1 - p)(p + (1 - p)^{\delta+1}) \\ &\leq np + n(1 - p)^{\delta+1} + np(1 - p) + n(1 - p)^{\delta+2} \\ &\leq np + n(1 - p)^{\delta+1} + np + n(1 - p)^{\delta+1} \\ &\leq 2np + 2n(1 - p)^{\delta+1} \\ &\leq 2np + 2ne^{-p(\delta+1)} \quad (\because 1 - p \leq e^{-p}) \end{aligned}$$

Hence, there is at least one choice of A such that $|A| + |B_A| + |C_A| \leq 2np + 2ne^{-p(\delta+1)}$. Also, the set $D_r = A \cup B_A \cup C_A$ is a restrained dominating set of G and $|D_r| = |A| + |B_A| + |C_A|$. Hence there exist at least one restrained dominating set, say D_r of G such that $|D_r| \leq 2np + 2ne^{-p(\delta+1)}$. Thus $\gamma_r(G) \leq 2np + 2ne^{-p(\delta+1)}$. This holds for any $p \in [0, 1]$.

Now to find the minimum value of the bound, we differentiate the right hand side with respect to p and set it equal to zero. The minimum value of right hand side will be obtained at $p = \frac{\ln(\delta+1)}{\delta+1}$. When we put this value of p in the upper bound, we get $\gamma_r(G) \leq \frac{2(1 + \ln(\delta + 1))}{\delta + 1}n$. \square

Now we present a randomized algorithm to find a restrained dominating set D_r of graph G . The algorithm is based on the probabilistic construction used in Theorem 9.2. The expectation of cardinality of restrained dominating set D_r returned by the following algorithm satisfies the upper bound of Theorem 9.2. Note that the algorithm can be implemented in linear-time.

Algorithm 2 RANDOMIZED-RESTRAINED-DOM-SET(G)

Input: A graph $G = (V, E)$.

Output: A restrained dominating set D_r of G .

begin

```
    Compute  $p = \frac{\ln(\delta+1)}{\delta+1}$ ;  
    Initialize  $A = B_A = C_A = \emptyset$ ;  
    foreach  $v \in V(G)$  do  
        | with the probability  $p$  decide if  $v \in A$  or  $v \notin A$ ;  
    foreach  $v \in V(G) \setminus N_G[A]$  do  
        |  $B_A = B_A \cup \{v\}$ ;  
    foreach  $v \in V(G) \setminus (A \cup B)$  do  
        | if  $N_G(v) \subseteq A \cup B$  then  
            | |  $C_A = C_A \cup \{v\}$ ;  
    Put  $D_r = A \cup B_A \cup C_A$ ;  
    return  $D_r$ ;
```

10 Conclusion

In this paper, we studied algorithmic aspects of the MRD problem. The RDD problem is already known to be NP-complete for chordal graphs, split graphs, planar graphs, undirected path graphs, and bipartite graphs. On the positive side, polynomial time algorithms are known to solve the MRD problem in trees and proper interval graphs. We proved that RDD problem is NP-complete for doubly chordal graphs, a subclass of chordal graphs, and proposed a polynomial time algorithm to solve the MRD problem in block graphs, a subclass of doubly chordal graphs. We also proposed algorithms to solve the MRD problem in threshold graphs (subclass of split graphs), cographs (superclass of threshold graphs), and chain graphs (subclass of bipartite graphs). We also observed that there exist graph classes for which domination and restrained domination differ in complexity. It is still interesting to look at the complexity status of the problem for other important subclasses of bipartite graphs and chordal graphs. In addition, We provided an upper bound on the restrained domination number of a graph in terms of number of vertices and degree of graph. We also proposed a randomized algorithm to compute a restrained dominating set of a graph, and proved that the cardinality of the restrained dominating set returned by our algorithm satisfies our upper bound with a positive probability.

References

- [1] A. Brandstädt, F.F. Dragan, V. Chepoi, V. Voloshin, Dually chordal graphs, SIAM Journal on Discrete Mathematics 11 (1998) 437-455.
- [2] A. Brandstädt, V. Chepoi, F.F. Dragan, The algorithmic use of hypertree structure and maximum neighbourhood orderings, Discrete Applied Mathematics 82 (1998) 43-77.
- [3] L. Chen, W. Zeng, C. Lub, NP-completeness and APX-completeness of restrained domination in graphs, Theoretical Computer Science 448 (2012) 1-8.

- [4] V. Chvátal, P. Hammer, Aggregation of inequalities in integer programming, Technical Report STAN-CS-75-518, Stanford University, California, 1975.
- [5] D. Corneil, H. Lerchs, L. Stewart-Burlingham, Complement reducible graphs, *Discrete Applied Mathematics* 3 (1981) 163-174.
- [6] D. Corneil, Y. Perl, L. Stewart, A linear recognition algorithm for cographs, *SIAM Journal on Computing* 14 (1985) 926-934.
- [7] R. Diestel. *Graph Theory*, volume 173. Springer, Berlin, fourth edition, 2010.
- [8] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, L.R. Markus, Restrained domination in graphs, *Discrete Mathematics* 203 (1999) 61-69.
- [9] G.S. Domke, J.H. Hattingh, M.A. Henning, L.R. Markus, Restrained domination in trees, *Discrete Mathematics* 211 (2000) 1-9.
- [10] D.R. Fulkerson, O.A. Gross, Incidence matrices and interval graphs, *Pacific Journal of Mathematics* 15 (1965) 835-855.
- [11] M.R. Garey, D.S. Johnson, *Computers and Interactability: a guide to the theory of NP-completeness*, Freeman, New York, 1979.
- [12] J.H. Hattingh, E. Jonck, E.J. Joubert, A.R. Plummer, Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs, *Discrete Mathematics* 308 (2008) 1080-1087.
- [13] J.H. Hattingh, E.J. Joubert, An upper bound for the restrained domination number of a graph with minimum degree at least two in terms of order and minimum degree, *Discrete Applied Mathematics* 157 (2009) 2846-2858.
- [14] J.H. Hattingh, E.J. Joubert, Restrained domination in claw-free graphs with minimum degree at least two, *Graphs and Combinatorics* 25 (2009) 693-706.
- [15] J.H. Hattingh, E.J. Joubert, Restrained domination in cubic graphs, *Journal of Combinatorial Optimization* 22 (2011) 166-179.
- [16] J.H. Hattingh, A.R. Plummer, A note on restrained domination in trees, *Ars Combinatoria* (94) (2010) 477-483.
- [17] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
- [18] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [19] M.A. Henning, Graphs with large restrained domination number, *Discrete Mathematics* 197/198 (1999) 415-429.
- [20] W. Hon, T. Kloks, H. H. Liu, S. Poon, Y. Wang, On Independence Domination, manuscript, 2013.
- [21] T. Kloks, D. Kratsch, H. Müller. Bandwidth of chain graphs, *Information Processing Letters*, 68 (1998) 313-315.

- [22] N. Mahadev, U. Peled, Threshold Graphs and Related Topics, in: Annals of Discrete Mathematics, vol. 56, North Holland, 1995.
- [23] B. S. Panda, D. Pradhan, A linear time algorithm to compute a minimum restrained dominating set in proper interval graphs, *Discrete Mathematics, Algorithms and Applications*, doi: 10.1142/S1793830915500202.
- [24] J.A. Telle, A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, *SIAM Journal on Discrete Mathematics* 10 (1997) 529-550.
- [25] M. Yannakakis, Node- and edge-deletion NP-complete problems. In Conference Record of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978), ACM, New York, (1978) 253-264.
- [26] B. Zelinka, Remarks on restrained domination and total restrained domination in graphs, *Czechoslovak Mathematical Journal* 55 (130) (2005) 393-396
- [27] V. Zverovich, A. Poghosyan, On Roman, Global And Restrained Domination in Graphs, *Graphs and Combinatorics* 27 (2011) 755-768